

KIRKMAN'S HYPOTHESIS REVISITED

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ABSTRACT. Watson proved Kirkman's hypothesis (partially solved by Cayley). Using Lagrange Inversion, we drastically shorten Watson's computations and generalize his results at the same time.

Kirkman's hypothesis [3] is (in changed notation) the formula

$$\begin{aligned} & \sum_{m=0}^M \sum_{n=0}^N \frac{1}{m+1} \binom{m+n}{n} \binom{2m+n+2}{m+n+2} \times \\ & \quad \times \frac{1}{M-m+1} \binom{M-m+N-n}{N-n} \binom{2(M-m)+N-n+2}{M-m+N-n+2} \\ & = \frac{2}{M+2} \binom{M+N+1}{N} \binom{2M+N+4}{M+N+4}. \end{aligned}$$

Kirkman could not prove it, but Cayley [1] proved the special case $N = 0$ in 1857. After more than hundred years, Watson [5] proved Kirkman's hypothesis by establishing the following power series expansions. Set

$$\psi(z, w) := \frac{1 - w - 2z - \sqrt{(1-w)^2 - 4z}}{2z(z+w)},$$

then

$$\begin{aligned} \psi(z, w) &= \sum_{m,n} \frac{1}{m+1} \binom{m+n}{n} \binom{2m+n+2}{m+n+2} z^m w^n, \\ \psi^2(z, w) &= \sum_{m,n} \frac{2}{m+2} \binom{m+n+1}{n} \binom{2m+n+4}{m+n+4} z^m w^n. \end{aligned}$$

Of course, Kirkman's hypothesis follows from this by writing $\psi \cdot \psi = \psi^2$ and comparing coefficients.

However, Watson's derivation of these two expansions required quite a bit of computation, in particular he treated both cases differently and separately.

Here, we present an extremely simple computation using the *Lagrange inversion formula* that has the advantage of not only treating both cases together but rather finding the power series expansion for $\psi^p(z, w)$ for general p . We refer for the Lagrange inversion formula to [2, 6].

The quadratic equation satisfied by $\psi(z, w)$ is

$$z(z+w)\psi^2(z, w) + (2z+w-1)\psi(z, w) + 1 = 0.$$

Date: February 1, 2008.

Writing $\psi = y/z$ and rearranging leads to the following equation of Lagrange type:

$$y = z \frac{(1+y)^2}{1-w(1+y)}.$$

With the Lagrange inversion formula we obtain:

$$\begin{aligned} [z^m w^n] \psi^p(z, w) &= [z^{m+p} w^n] y^p(z, w) = \frac{p}{m+p} [y^m w^n] \left(\frac{(1+y)^2}{1-w(1+y)} \right)^{m+p} \\ &= \frac{p}{m+p} \binom{m+n+p-1}{n} [y^m] (1+y)^{2m+n+2p} \\ &= \frac{p}{m+p} \binom{m+n+p-1}{n} \binom{2m+n+2p}{m+n+2p}. \end{aligned}$$

This leads with $\psi^r \cdot \psi^s = \psi^p$ to the convolution formula (*generalized Kirkman hypothesis*):

$$\begin{aligned} &\sum_{m=0}^M \sum_{n=0}^N \frac{r}{m+r} \binom{m+n+r-1}{n} \binom{2m+n+2r}{m+n+2r} \times \\ &\quad \times \frac{s}{M-m+s} \binom{M-m+N-n+s-1}{N-n} \binom{2(M-m)+N-n+2s}{M-m+N-n+2s} \\ &= \frac{p}{M+p} \binom{M+N+p-1}{N} \binom{2M+N+2p}{M+N+2p}. \end{aligned}$$

For other results of Kirkman's, treated with the Lagrange inversion formula, see [4, Ex. 6.33-c].

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